THE SECOND DERIVATIVE TEST: A CASE STUDY OF INSTRUCTOR GESTURE USE

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We present a case study of how five instructors used gesture when introducing the second derivative test in a first semester calculus class. The second derivative test and optimization naturally evoke hand motions while teaching, making this a fertile ground for studying gesture use in the classroom. Each of the five instructors used a classic optimization problem as a primary example of how one would use the second derivative test to verify that a found value would be a maximum or minimum. We observed the instructors making connections between the algebraic, numerical, and graphical representations of this concept through gesture. The quantity and types of gesture varied greatly by instructor, but there were two key links that every instructor made.

Keywords: Instructional Activities and Practices, Classroom Discourse, Post-Secondary Education

Introduction

There is a large body of work in cognitive science focused on embodied cognition; our knowledge is shaped by our experiences and interactions with the world around us (Lakoff & Nunez, 2000; Nemirovsky & Ferrara, 2009; Nemirovsky, Tierney, & Wright, 1998). Through bodily experiences, such as gesture, our understanding of complex concepts is shaped. As calculus can be considered the study of motion, it is a natural place to examine gesture. There are several problems that require students to visualize/imagine situations involving rate(s) of change – related rates and optimization are two examples. Diagrams and graphs are means of visualization that may facilitate the understanding of many concepts and problems in calculus. While a student may read an example in the textbook and model a solution based on this text, an example presented in class provides the learner an additional modality for learning: the instructor's gestures. An instructor's gestures can give life to a static diagram or a point on a graph. There is a growing body of research that suggests that an instructor's use of gesture can have a positive impact on student learning (Alibali & Nathan, 2007; Hostetter & Alibali, 2010).

Gesture use has recently become the focus of many educational studies. While the literature related to gestures made in undergraduate mathematics classroom is limited (Marrongelle, 2007; Rasmussen, Stephan, & Allen, 2004; Wittmann, Flood, & Black, 2013; Yoon, Thomas, & Dreyfus, 2011), there is a plethora of studies examining gesture in K-12 classrooms (Alibali & Kita, 2010; Arzarello, Paoloa, Robutti, & Sabena, 2009; Cook, Duffy, & Fenn, 2013; Cook, Mitchell, & Goldin-Meadow, 2008; Edwards, 2009; Goldin-Meadow, Kim, & Singer, 1999; Maschietto & Bussi, 2009; Nemirovsky et al., 1998; Roth & Thom, 2009). Goldin-Meadow (2000) stated, "[a] task for the future is to determine how gesture can best be harnessed to improve communication in classrooms" (p. 235) and Roth and Lawless (2002) noted, "little is known about the role of gesture in learning and instruction" (p. 285).

Gesture use in the calculus classroom has not been extensively studied. We seek to address this gap in the literature by examining how instructors use gesture in natural teaching environments. In this paper, we address the questions: What examples does an instructor use to communicate mathematical ideas and concepts? While working through the chosen examples, what key ideas are linked? What gestures does the instructor make while making links for students? This study examines the answers to these questions at one point in time – when the instructor introduces the second derivative test.

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Theoretical Perspective

Mathematics instructors often make connections between ideas and concepts. In this research, we focus on what Alibali et al. (2014) defined as a linking episode: segments of discourse in which teachers connect ideas. Note that a linking episode may contain several distinct links. For example, we observed instructors making two distinct links in a single linking episode. We examined the linking episodes that occurred naturally during calculus instruction and the extent to which instructors used varying modalities, particularly gesture, to express links between ideas.

Gestures are a natural part of communication and may convey additional information to the listener to foster comprehension. Yoon et al. (2011, pp. 891-892) indicate "gestures are a useful, generative, but potentially undertapped resource for leveraging new insights in advanced levels of mathematics" (p. 891-892). They further advise that instructors should model gestures for students in lecture. Hence, we want to observe how instructors use gesture to highlight the ideas that they are connecting during a linking episode.

Methods

A qualitative case study methodology (Cohen, Manion, & Morrison, 2011) was used to examine how five instructors used gesture during natural classroom activities that involved the second derivative test. Each lesson in which the second derivative test was introduced was transcribed and broken down into linking episodes as described by Alibali et al. (2014). These linking episodes were then analyzed to determine how many and what types of gesture were used in conjunction with the links being made. Links were coded as multi-modal if they were made using some combination of speech, writing, and gesture, and were coded as uni-modal if they were made in speech alone. In the transcription, speech made in conjunction with gesture are in bracketed pairs: [speech uttered][gesture description].

For this study, gestures were coded according to the static and dynamic categorizations defined by Garcia and Engelke Infante (2012, 2013) and the pointing and writing gestures defined by Alibali et al (2014). Engelke Infante and Garcia examined how students use gesture when solving related rates and optimization problems in first semester calculus and identified two broad categories of gesture use: dynamic and static (Garcia & Engelke Infante, 2012, 2013). Dynamic gestures consist of moving the hands to describe action that occurs in the problem or movements made to represent mathematical concepts. Within dynamic gestures two subcategories were identified: dynamic gestures related to the problem (DRP) and gestures that are not related to the problem (DNRP). We added a third subcategory to dynamic gestures, Dynamic Algebraic. Dynamic Algebraic (DA) gestures are made to indicate how we manipulate quantities in an equation. For example, we often speak of "moving *x* to the other side" and make an under/over swooping motion with a hand.

Static gestures are made to illustrate a fixed value (length, radius, volume, etc.) or to illustrate properties of a geometric object. Note that while the term static is used, that does not necessarily mean that there is no motion involved. Again, the category of static gestures was further sub-categorized into static gestures related to a fixed value (SFV) and gestures related to the shape of a graph (SSG). SFV and SSG are both primarily types of iconic gestures but may also be metaphoric. Garcia and Engelke Infante (2012, 2013) observed static gestures primarily being used to facilitate diagram construction.

As defined by Alibali et al. (2014), pointing gestures are those used to index objects, locations, and inscriptions in the physical world. Writing gestures are those in which writing or drawing is integrated with speech in a way in which a writing instrument is used to indicate content of the accompanying speech (such as drawing a circle around a term in an equation while stating 'this term'). We further categorized points as either static points (SP-pointing to one static object on the board, such as a point on a graph), tracing points (TP-using a finger, hand or other body part to trace the shape of an object on the board, such as a finger tracing along a tangent line), or generic points

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(GP-points directing attention to something that was done before, but not a 'look at this particular thing' type of point).

Results

Here we examine how the five instructors introduced the second derivative test. In each case, the second derivative test was introduced in conjunction with the solution to an optimization problem. All of the instructors used a similar example problem to illustrate the second derivative test – construct a fence that maximizes area for a given perimeter (Instructors A, B, and C) or construct a fence of minimum perimeter for a fixed area (Instructors D and E). Instructors A, B and C introduced the second derivative test on the first day of instruction on optimization while the other two instructors introduced the second derivative test on the second day of instruction on optimization.

We observed that there were two key links made by every instructor. First, every instructor made explicit the link between the sign of the second derivative and the concavity. Second, every instructor made a link between the concavity (shape of the graph) and the optimizing function having a maximum or minimum value. Often, these two links occurred in succession as a single linking episode (see rows two and three of the table). Table 1 summarizes how these key links were made by each instructor and indicates whether the link was accompanied by gesture. Observe that Instructors A, B, and C made all of their links multi-modally while Instructors D and E made about half of their links multi-modally.

Table 1: Links made by calculus instructors while explaining the second derivative test

Linking Episode	Representations Linked*	A	B	C	D	E
f'' – concavity	A-G	1+ (SP - SSG)				
f''negative – CCD – max	A-G-N	2+ (SSG-SP)	2+ (SSG-WG-SFV)	2+ (SFV-WG-RP)	1+ (GP)	
f''positive – CCU – min	A-G-N	1+ (SSG-SP)			1+ (SP)	1+ (SP)
CCD – max	G-N	1+ (TP-SP)	1+ (SFV-TP)			
CCU – min	G-N	1+ (TP-SP)		1+ (SSG-SFV)		1
$f' = 0, f'' > 0$ – min	G-N				1	
$f' = 0, f'' < 0$ – max	G-N				1	

*A = algebraic, G = graphical, N = numerical, + = (gestures used); SP=static point, SSG=static shape of graph, SFV=static fixed value, TP = tracing point, WG = writing

In the interest of space, we will examine the differences in how Instructors A, B and C used gesture to highlight key information in their linking episodes. Instructor A presented the second derivative test before starting her example (about 3 minutes), and then reiterated it during her example problem (about 2 minutes). Instructors B and C both explained the second derivative test completely within the context of the example and used 2-3 minutes of class time.

Instructor A

During Instructor A's brief overview of the second derivative test, she drew a generic graph with a maximum and minimum value. As she pointed to the maximum value on her graph, she stated "we know we have [a maximum] [SP]," and then she pointed to the minimum on her graph [on the far right in Figure 1] and stated "we know we have [a minimum] [SP], but we don't know this starting out.” She reviewed what a critical value was and asked the students “what does the second derivative tell us?” The students responded, “concavity.” She proceeded to make a static point to the maximum on the graph and asked “[what’s the concavity right here?] [SFV-graph on far right of Figure 1]” while making a circular motion with her hand around the maximum on the graph. This led to the conclusion that a concave down graph would have a maximum which was pointed to again. Similarly, the class arrived at the idea that a concave up graph would have a minimum. It was then indicated that they would dig into this idea in the next example. Here, we observe Instructor A making explicit the second key link between the concavity of the graph and the existence of a maximum (or minimum) and that she highlights these features on the graph with static gestures.

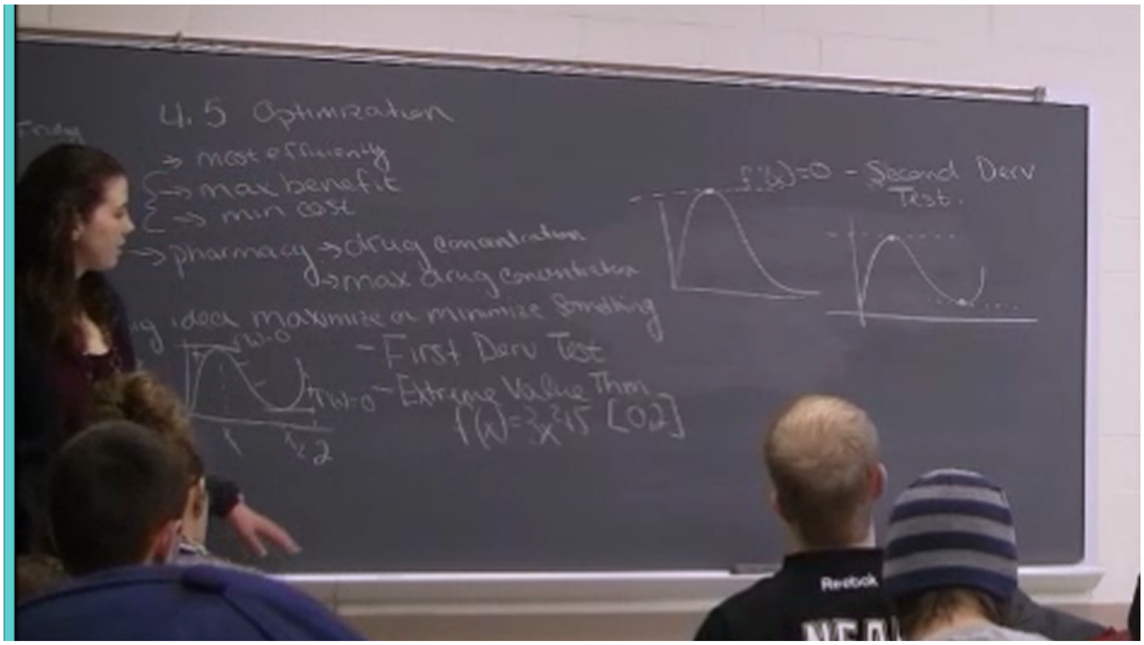

Figure 1. Instructor A’s overview of the second derivative test.

The first example of an optimization problem presented to the class immediately following the discussion of the second derivative test was a problem involving a farmer who wishes to maximize a rectangular area along a river with a given amount of fence. Upon getting to the point in the problem where the maximum needs to be verified, the instructor discussed what the domain of the problem was and in doing so, made several tracing point gestures in reference to her diagram. She then stated, “We could use either the first or second derivative test, but since we haven’t seen the second derivative test in action, let’s do that.” She continued, “So, second derivative, I’m going to look [at] [SP -to the first derivative function] the first derivative, right? So, the derivative of fifteen hundred is nothing and that’s negative 4 (writes equation for second derivative on the board).” Her next statement included the first hand gestures relating the shape of the graph and the second derivative test. She stated:

A: So, I have either [a peak][SSG-made a small, cupped hand gesture mimicking the shape of a concave down graph] or [a valley][SSG-made a small, upward swooping hand gesture mimicking the shape of a concave up graph] because the derivative is zero. What does the [second derivative negative mean?][SP-to second derivative function] Concave up or concave down? [Down,] [WG-draws downward parabola shape] so it looks like that. Does that give you a max or min?

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Here, we observe Instructor A making explicit both key links with gesture (primarily pointing gestures) in a single linking episode.

Instructor C

The first example of how to solve an optimization problem presented by Instructor C involved finding the dimensions of the largest possible rectangular shaped vegetable garden with a fixed perimeter. As he neared the end of the problem, he introduced the second derivative test. He verbally explained that they computed the value for the second derivative and found it to be negative. At this point, he exclaims, “The second derivative is [ne-ga-tive] [SRF-arms up and out and pointing down with fingers] What does that mean the function is doing? Second derivative. Concave? [Down!] [SRF-points arm and finger down].” Hence, he has made explicit the first key link between the sign of the second derivative and the shape of the graph (concave down).

Next, Instructor C indicated, “When x is 40, the graph is concave down. If you draw just anything that’s [concave down] [WG-draws concave down graph on board and makes point at local max] is the point a [maximum point?] [SP-points at the point made on graph].” Here, he made explicit the second key link between the concave down shape of the graph and the function having a maximum value.

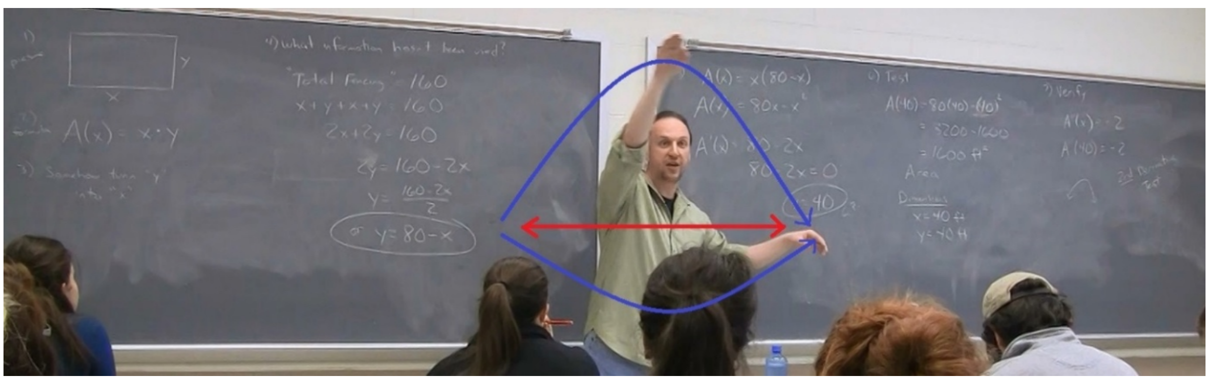

Figure 2. Instructor C’s gestures to illustrate the second derivative test.

Instructor C then moved to the center of the classroom and exclaimed, “Watch! Watch! Watch!” and made sure that all eyes were on him before he proceeded to summarize the second derivative test with a sequence of large gestures (Figure 2). He demonstrated that a concave down graph would have a maximum value (blue arching motion above the red line) while a concave up graph (blue arching motion below the red line) must have a minimum value. He related each of these ideas back to the idea of a horizontal tangent line (which he indicated with the horizontal red line motion). This was the only instance we observed where the gestures were made as a deliberate part of the lesson.

Instructor B

Here we focus on the distinctly different way in which Instructor B makes the second key link between the concavity and the optimizing function having a maximum. She began by drawing a concave down graph on the board and making a small, one-handed concave down motion like Instructor C, asking whether there would be a maximum or minimum. She then proceeded to point to the maximum on the graph while stating that it was the maximum of the graph (Figure 3a). She took one step to the side of the graph and reiterated that a concave down graph could have a maximum while making the small, one-handed gesture again. Next, she took another step to the side and made a very large two-handed gesture while stating that since it was concave down everywhere, the value would have to be an absolute maximum (Figure 3b) followed by stepping back to her graph on the board and pointing to the maximum again.

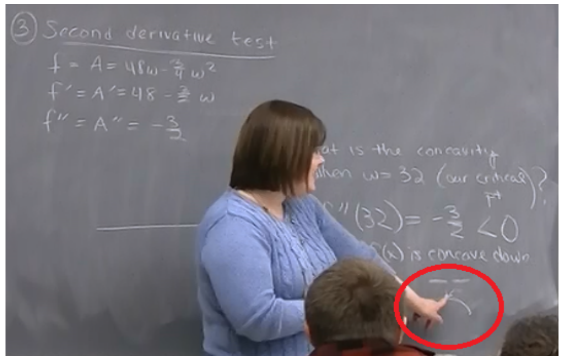

Figure 3a. Pointing to the maximum on a concave down graph.

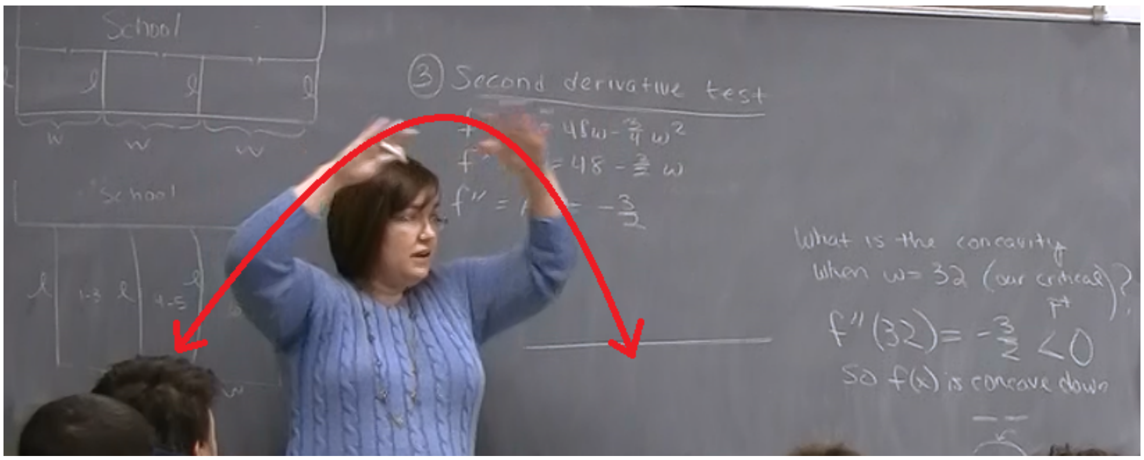

Figure 3b. Large, two-handed gesture for a concave down graph.

While making the second key link, Instructor B was very animated in her use of gesture to make sure students made the connection between the graph being concave down and the function having a maximum value. In fact, during this one linking episode she made the link three times, each of which was multi-modal. Instructor B used a combination of points at the board with large gestures in the space in front her.

Similar to the linking episodes observed by Alibali and colleagues in middle school teachers (Alibali et al., 2014), these instructors were linking several mathematical ideas for their students. Each instructor linked ideas about the second derivative function, the concavity/shapes of graphs, and maximum/minimum values during their lessons. Hence, we conclude that any linking episode about the second derivative test should include the link between the sign of the second derivative and the concavity and the link between the concavity (shape of the graph) and the optimizing function having a maximum or minimum value. However, from Table 1, we know that the manner in which these links are made can vary greatly. Indeed, Instructor D, summarized these key links uni-modally (in speech alone) by saying: "The first derivative test tells us that if *f* prime is equal to zero and *f* double prime is positive, we have to have a local min. If *f* double prime, if *f* prime is zero, *f* double prime is negative, we have to have a local max. If they're both equal to zero, we don't know." Note that these verbal statements fail to capture the link between concavity (shape of the graph) and the existence of a max/min.

Conclusion

While each of the instructors used a similar example problem to introduce the second derivative test, there was significant variation in the linking episodes used to convey the key ideas of the

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concept. As other studies have indicated, we observed that instructors frequently used gesture when referring to a graph or diagram to illuminate key features of the object. These findings support theories about connections between diagrams and gesture presented by de Freitas and Sinclair (2011).

As observed by (Alibali et al., 2014), instructors were often making links between the various representations of the concept. Four of the five instructors used a gesture that was depictive of a concave down (up) graph while emphasizing with words that a concave down graph would have a maximum (minimum) value. Some instructors went beyond this, as Instructor B did, and pointed at least once to the maximum on the graph that she had drawn.

Not all instructors used the same quantity or distribution of gestures. However, quantity is not the only measure of gestures usefulness. An appropriately timed gesture of a particular type may be equally important. For example, Instructor C made certain that his students would be paying attention to at least one sequence of gestures he made. Future research will employ a design experiment in which we will examine the impact of particular gestures used during instruction on student learning. We will assess students' learning based on whether they learned from an instructor who deliberately incorporated particular gestures or an instructor who did not purposefully incorporate gesture into the lesson. In time, we expect to create an online catalogue of lessons to showcase how the deliberate incorporation of gestures can enable instructors to foster greater understanding of calculus concepts in their students.

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